

Renormalization group in the internal space

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Renormalization group in the internal space consists of the gradual change of the coupling constants. Functional evolution equations corresponding to the change of the mass or the coupling constant are presented in the framework of a scalar model. The evolution in the mass which yields the functional generalization of the Callan-Symanzik equation for the one-particle irreducible effective action is given in its renormalized, cutoff-independent form. The evolution of the coupling constant generates an evolution equation for the two-particle irreducible effective action.

I. INTRODUCTION

This paper presents a modified version of the functional renormalization group (RG) method aiming at the two-particle irreducible (2PI) Green functions. The renormalization group strategy consists of the monitoring Green functions for the elementary field variables as the fluctuation modes are turned on gradually [1]. There are two further ingredients which extend this method into a general purpose, non-perturbative algorithm to solve quantum field theories. The first is the functional generalization of the blocking where an infinitesimal fraction of the modes are turned on or off [2, 3, 4, 5]. This reflects the following important change of strategy. In the original version of the renormalization group method a finite fraction of the modes are turned on or off and one relies on the perturbation expansion in obtaining the change in the dynamics. In order to obtain reliable results, one should work in high enough order of the perturbation expansion using the few bare, relevant vertices. The method of using infinitesimal blocking steps is based on another systematical approximation where infinitely many vertices are used in a one-loop renormalization group equation. The manipulation of a large number of constants can be carried out by means of their generating functional. This is the reason of looking for the functional generalization of the renormalization group scheme. The evolution equation which looks like a one-loop equation is actually exact as long as it is solved in an infinite dimensional functional space. Since the general theory of functional differential equations is not well developed we have to rely on a truncation of the evolution equation, the only approximation of the method. It is based on the projection of the functional differential equation into a functional subspace. The tailoring of this subspace seems to be an interesting and a physically better motivated approximation scheme than the truncation of the perturbative series.

In order to motivate another modification of the original strategy we recall that the order in which the modes are eliminated in the renormalization group method,

$$e^{-\frac{1}{\hbar}S_{n-1}(\phi_1, \dots, \phi_{n-1})} = \int d\phi_n e^{-\frac{1}{\hbar}S_n(\phi_1, \dots, \phi_n)}, \quad (1)$$

is usually given by means of a scale parameter in the external space of field theory, the space-time, such as the momentum or energy and the dynamics is studied as the function of the UV or IR cutoff. In this manner the short distance modes whose dynamics is rendered more perturbative by the kinetic energy are eliminated first and the more problematical nonperturbative modes are treated later when the blocked action is sufficiently renormalized, $p_1 \leq p_2 \leq \dots \leq p_{n-1} \leq p_n$. An alternative to this procedure is the ordering of the modes according to their scale in the internal space, their amplitude. The sharp cutoff version of this procedure consists of the study of the λ -dependence of the path integral

$$Z_\lambda = \int_{-\lambda}^{\lambda} d\phi_1 \dots \int_{-\lambda}^{\lambda} d\phi_n e^{-\frac{1}{\hbar}S_n(\phi_1, \dots, \phi_n)}. \quad (2)$$

A more natural, smooth cutoff realization of this strategy is the functional generalization of the Callan-Symanzik equation [6, 7] where the mass is evolved in a relativistic theory [8, 9],

$$Z = \int D[\phi] e^{-\frac{1}{\hbar}S[\phi] - \frac{\lambda}{2\hbar} \int_x \phi_x^2}. \quad (3)$$

Since the mass term dominates the dynamics of free fluctuations whose wavelength is longer than the inverse mass the two procedures produce similar scale dependences.

There are in the same time essential differences between the external and the internal space realizations of the renormalization group. The most important one is that the physics is kept fixed along the renormalized trajectory of the external space method but changes together with the control parameter which suppresses the fluctuations. This difference arises from the different ways the UV modes are handled during the evolution. In principle both methods are similar, their evolution equations look formally equivalent, only the actual form of the suppression of the modes is different. We may introduce a straightforward division line between these methods by stating that a suppression mechanism corresponds to the external or internal space scheme if the corresponding evolution equation is UV finite or divergent. The internal space methods require the introduction of an UV regulator which is kept fixed during the evolution and the final products of the evolution are the bare Green functions, as opposed to the external space methods where the gliding cutoff renormalizes the Green functions automatically.

In the first part of this paper we present a modification of the internal space evolution equation which eliminates this shortcoming and yields UV finite, renormalized Green functions. This is achieved by the functional implementation of a renormalization which is reminiscent of the BPHZ method [10, 11, 12]. One finds as byproduct a possibility of a Hartree-type optimization of the unavoidable truncation of the evolution equation. These issues are studied in the framework of the effective action for the field variable, the generating functional of the one-particle irreducible (1PI) Green functions and serve as a preparation for the second part of this work, the construction of an internal space renormalization scheme for the computation of the 2PI Green functions.

This step is made by generalizing the Callan-Symanzik scheme into a far more flexible procedure by allowing the control parameter, the mass square in the preceding case, to be replaced by *any* parameter of the dynamics which controls the physics in a differentiable manner. Such kind of an extension is needed when symmetry considerations require the presence of soft modes and the suppression of the fluctuations by a mass term is inconsistent. Actually, the mass controls the *amplitude* of the fluctuations but disregards their *nature*. The Gaussian fluctuations in the absence of interactions are rather peculiar, their higher moments factories according to Wick's theorem. The interactions introduce non-factorisable correlations.

It is easy to identify a control parameter which turns the non-factorisable fluctuations on. In fact, it is just the coupling constant or a common multiplicative factor in front of all coupling constants if there are several of them. The evolution equation, generated by the coupling constant is highly involved [8]. In fact, the evolution equation contains one-loop integrals as long as the suppression term which generates the evolution is quadratic in the field. The complications arising from a quartic suppression term, ϕ^4 , the two- and three-loop integrals in the evolution equation reflect the complicated structure of excitations created by the operator ϕ^4 in terms of single-particle excitations. One can simplify the situation, at least formally, by using composite operators in which the suppression term is again quadratic. The evolution equation for the effective action for such composite fields will be as simple as that of the Callan-Symanzik scheme. In the case of the ϕ^4 scalar theory the composite operator one turns at is obviously ϕ^2 and one expects that the gradual turning on the interactions should be possible to realize in a simple manner by following the evolution of the effective action for the composite field ϕ^2 . But the physics encapsulated by such an effective action is far more restricted than those found in the effective action for the non-local composite operator $\phi_x\phi_y$ [13, 14]. In fact, the polarization cloud around a bare particle requires the possibility of separating the two local, elementary field variables in ϕ^2 in order to trace the space-time structure of dressed, physical excitations. Therefore our proposal, spelled out in the second part is to describe the structure of vacuum up to two-particle correlations by solving the evolution equation for the effective action for the propagator.

This scheme is useful if the initial conditions, the effective action at the initial value of the coupling constant is known. One should therefore give the initial conditions for vanishing or infinitesimally weak coupling strength. The evolution equation will then be integrated until the physical value of the coupling constant is reached. We shall investigate in this work the solution arising from the non-interacting initial condition only. Such a scheme has already been found useful in performing gauge-free computations in QED [15], in constructing systematically the density functional [16], and for identifying the localization-localization transition for disordered systems [17].

There are formal similarities between the Schwinger-Dyson hierarchy of equations and the evolution equation. But the main advantage of the evolution equation approach is that it provides a coupled set of equations for the Green functions of the composite field which are uniquely solvable for a given initial condition in the perturbative regime.

The organization of the paper is the following. In Sect. II a short description of the functional Callan-Symanzik approach is given, its solution by iteration and the improvement of the truncated iteration by optimization is pointed out. Sect. III presents the derivation of the internal space RG evolution equation for the 2PI effective action for a family of composite fields, the bilinear expressions of the elementary field. The RG equations for the effective theory are presented in Sect. IIIB after expanding the 2PI effective action around the vacuum. Instead of their numerical solution an optimized iterative approach for solving the evolution equation of the 2PI effective action is discussed in Sect. IIIC, and the analytic solution found truncating the iteration just after the first step. The effective theory for the composite field is related to the theory for the elementary field in Sect. IV. Finally, the results are summarized in Sect. V.

II. 1PI EFFECTIVE ACTION

The simplest realization of the internal space renormalization group method is the control of the amplitude of the fluctuations by the mass term which yields a functional generalization of the Callan-Symanzik equation. As the simplest example we consider the Euclidean scalar model defined by the bare action

$$S[\varphi] = \int_x \left[-\frac{1}{2} \varphi_x \square_x \varphi_x + \frac{1}{2} \mu^2 \varphi_x^2 + \frac{g}{4!} \varphi_x^4 \right], \quad (4)$$

see Appendix A for the notations and conventions. The generating functional

$$e^{\frac{1}{\hbar} W[j]} = \int \mathcal{D}[\varphi] e^{-\frac{1}{\hbar} S[\varphi] + \frac{1}{\hbar} \varphi \cdot j} \quad (5)$$

is given by a regulated path integral where the UV regulator is not shown explicitly. It gives rise the effective action

$$\Gamma[\phi] = -W[j] + j \cdot \phi, \quad \frac{\delta W[j]}{\delta j} = \phi \quad (6)$$

satisfying the relations

$$j = \frac{\delta \Gamma[\phi]}{\delta \phi}, \quad \frac{\delta^2 W[j]}{\delta j \delta j} \cdot \frac{\delta^2 \Gamma[\phi]}{\delta \phi \delta \phi} = 1. \quad (7)$$

A. Evolution equation

The dependence of $W[j]$ and $\Gamma[\phi]$ in the mass μ^2 , considered further on as our control parameter, is related,

$$\partial_{\mu^2} \Gamma[\phi] = -\partial_{\mu^2} W[j], \quad (8)$$

and the evolution equation can be obtained by simply bringing the derivative ∂_{μ^2} into the functional integration in Eq. (5),

$$\partial_{\mu^2} \Gamma[\phi] = \frac{1}{2} \int_x \left[\hbar \frac{\delta^2 W[j]}{\delta j_x \delta j_x} + \left(\frac{\delta W[j]}{\delta j_x} \right)^2 \right] = \frac{1}{2} \int_x \left[\hbar \left(\frac{\delta^2 \Gamma[\phi]}{\delta \phi \delta \phi} \right)_{x,x}^{-1} + \phi_x^2 \right]. \quad (9)$$

It is worthwhile mentioning that the harmless looking step of bringing the partial derivation with respect to the control parameter of the evolution within the path integral produces in general highly non-trivial results. First, it can be considered as the generalization of the Hellmann-Feynman theorem [18, 19] for time-dependent processes. Second, different choices for the control parameter λ produce important, well-known equations: (i) When $\lambda = \varphi_x$, the functional derivative with respect to one of the integral variables is vanishing. The stability of the path integral against the shift of the integral variables produces the Schwinger-Dyson hierarchical equations (c.f. App. D1). (ii) If λ is a parameter of a transformation of φ_x which preserves at least a part of the action invariant then the results are the Ward identities.

Here we use the simplest realization of the RG idea to control the amplitude of the field fluctuations, namely the control is achieved via the mass term $\lambda = \mu^2$ in a momentum-independent manner [8] in order to stay as close as possible to the discussion of the evolution equation for the 2PI effective action presented below. There are other choices which allow one to control the amplitude. The most widely used suppression methods imply strong momentum dependence [2, 3, 4, 5].

The graphical interpretation of Eq. (9) is obtained by following the Callan-Symanzik strategy. In fact, the μ^2 -dependence of a Feynman graph F with legs amputated arises from its internal lines, the propagators G ,

$$\partial_{\mu^2} F = \sum_p \frac{\delta F}{\delta G_p} \partial_{\mu^2} G_p = - \sum_p \frac{\delta F}{\delta G_p} S_p^{(2)-1} \partial_{\mu^2} S_{\mu^2,p}^{(2)} S_p^{(2)-1}. \quad (10)$$

The evolution equation is the application of this chain rule for the 1PI graphs in the framework of the skeleton expansion.

Contrary to the strategy of the Callan-Symanzik equation where one expands around the massless, critical system our initial condition is supposed to be in the perturbative regime. Therefore we choose $\mu_0^2 \gg \Lambda^2$ where Λ denotes the

UV cutoff as initial condition because the quantum fluctuations are suppressed and the tree-level approximation to the effective action,

$$\Gamma_{\text{in}}[\phi] = \int_x \left[-\frac{1}{2} \phi_x \square_x \phi_x + \frac{\mu_0^2}{2} \phi_x^2 + \frac{g}{4!} \phi_x^4 \right] \quad (11)$$

is applicable.

B. Renormalization

There are two related reasons to perform further adjustments on the evolution equation (9). One is that the control parameter used in the internal space evolution does not regulate the UV divergences. As a result Eq. (9) has to be first regulated and the resulting effective action must be renormalized in order to extract Green functions which correspond to cutoff independent scales. Another, more technical looking complication is that the integration of the evolution equation (9), starting with the initial condition (11), generates more and more complicated ϕ -dependence in the effective action as μ^2 is decreased. The two issues are related because once the effective action is expressed in terms of the renormalized quantities it is supposed to contain slower, i.e. easier to truncate dependence in the field.

The unavoidable truncation of the evolution equation should be as harmless as possible from the point of view of the terms appearing in the inverse of the right hand side of the evolution equation (9). There are two different directions to follow in the search of optimization. One is the appropriate choice of the dependence on the control parameter of the evolution in the Lagrangian. Because all we want to achieve is an interpolation between a simple, calculable initial condition and the true system only, different possibilities might be available to introduce the control parameter. This avenue has already been explored in the context of optimizing the blocking transformation in the traditional realization of the renormalization group method in external space [20, 21]. We cannot follow this direction here, having already chosen the mass as control parameter because of other reason. The other dimension of the optimization is the tailoring of the functional form of the effective action. One should recall in this respect that the magnitude of the variable ϕ has nothing to do with the strength of the quantum fluctuations since ϕ is a book-keeping device only in defining the generating functional for the 1PI Green functions. Nevertheless ϕ is supposed to be small in an indirect manner for an optimized scheme because the more complicated terms of the effective action tend to be of higher order in ϕ . More precisely, the omission of the terms $\mathcal{O}(\phi^{2n})$ leads to ignoring the n -particle correlations.

We shall try to slow down the ϕ -dependence of the effective action by the appropriate truncation of the inverse on the right hand side of the evolution equation. The inverse will be written as a series where terms appear in the order of increasing complexity in such a manner that the more complicated terms are as small as possible. The tailoring of the "free" inverse propagator around which the inverse is expanded is reminiscent of the Hartree approximation when one adds a mass term to the free Lagrangian, subtracts it from the interaction part, and optimizes the perturbation expansion with respect to this rearrangement of the Lagrangian. But one should notice that the same strategy is followed in establishing the renormalized perturbative series, too. Therefore we separate off local counterterms, $\Gamma_{CT}[\phi]$, from the effective action,

$$\Gamma[\phi] = \Gamma_R[\phi] + \Gamma_{CT}[\phi] \quad (12)$$

in such a manner that the renormalized effective action, $\Gamma_R[\phi]$, converges as the cutoff is removed. While this separation has no effect on the exact solution, it will influence the solution of the truncated evolution equation. The condition $\Lambda \ll \mu_0^2$ clearly prevents us to send the cutoff to infinity. But an approximate cutoff independence of $\Gamma_R[\phi]$ can still be tested for $\mu^2 \ll \Lambda^2 \ll \mu_0^2$ for a sufficiently large but finite μ_0^2 .

The functional Taylor expansion involving the renormalized 1PI Green functions $\Gamma_{n \ x_1, \dots, x_n}$ and their counterterms, $\Gamma_{CT \ n \ x_1, \dots, x_n}$, will be used,

$$\Gamma_R[\phi] = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \Gamma_{2n \ x_1, \dots, x_{2n}} \phi_{x_1} \cdots \phi_{x_{2n}}, \quad \Gamma_{CT}[\phi] = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \Gamma_{CT \ 2n \ x_1, \dots, x_{2n}} \phi_{x_1} \cdots \phi_{x_{2n}}, \quad (13)$$

where double indices are integrated over, c.f. Appendix A. The evolution equation

$$\partial_{\mu^2} \Gamma_R[\phi] + \partial_{\mu^2} \Gamma_{CT}[\phi] = \frac{\hbar}{2} \text{Tr} \left(\frac{\delta^2 \Gamma_R[\phi]}{\delta \phi \delta \phi} + \frac{\delta^2 \Gamma_{CT}[\phi]}{\delta \phi \delta \phi} \right)^{-1} + \frac{1}{2} \phi \cdot \phi \quad (14)$$

is then written by means of expanding the inverse on the right hand side around the renormalized propagator, $G_R = \Gamma_2^{-1}$, and by splitting the resulting series into the sum of two pieces in d dimensions,

$$\partial_{\mu^2} \Gamma_R[\phi] = \frac{\hbar}{2} \sum_{m=[\frac{d}{2}]+1}^{\infty} (-1)^m \text{Tr} \left[G_R \cdot \left(\sum_{n=2}^{\infty} \frac{1}{(2n-2)!} \Gamma_{2n \ x_3, \dots, x_{2n}} \phi_{x_3} \cdots \phi_{x_{2n}} \cdot G_R \right)^m \right] + \frac{1}{2} \phi \cdot \phi, \quad (15)$$

and

$$\begin{aligned} \partial_{\mu^2} \Gamma_{CT}[\phi] &= \frac{\hbar}{2} \text{Tr} \left(\frac{\delta^2 \Gamma_R[\phi]}{\delta \phi \delta \phi} + \frac{\delta^2 \Gamma_{CT}[\phi]}{\delta \phi \delta \phi} \right)^{-1} \\ &\quad - \frac{\hbar}{2} \sum_{m=[\frac{d}{2}]+1}^{\infty} (-1)^m \text{Tr} \left[G_R \cdot \left(\sum_{n=2}^{\infty} \frac{1}{(2n-2)!} \Gamma_{2n \ x_3, \dots, x_{2n}} \phi_{x_3} \cdots \phi_{x_{2n}} \cdot G_R \right)^m \right], \end{aligned} \quad (16)$$

where the first two space-time indices of Γ_n were suppressed. This split is made in such a manner that the renormalized part remains "finite", i.e. approximately cutoff independent at energy scales p as long as $p^2 \ll \Lambda^2 \ll \mu_0^2$. The expansion of the inverse on the right hand side of Eq. (16) around G_R ,

$$\begin{aligned} \partial_{\mu^2} \Gamma_{CT}[\phi] &= \frac{\hbar}{2} \sum_{m=0}^{\infty} (-1)^m \text{Tr} \left\{ G_R \left[\left(\Gamma_{CT \ 2} + \sum_{n=2}^{\infty} \frac{1}{(2n-2)!} (\Gamma_{2n \ x_3, \dots, x_{2n}} + \Gamma_{CT \ 2n \ x_3, \dots, x_{2n}}) \phi_{x_3} \cdots \phi_{x_{2n}} \right) G_R \right]^m \right\} \\ &\quad - \frac{\hbar}{2} \sum_{m=[\frac{d}{2}]+1}^{\infty} (-1)^m \text{Tr} \left\{ G_R \left[\sum_{n=2}^{\infty} \frac{1}{(2n-2)!} \Gamma_{2n \ x_3, \dots, x_{2n}} \phi_{x_3} \cdots \phi_{x_{2n}} G_R \right]^m \right\}, \end{aligned} \quad (17)$$

shows that each contribution to $\Gamma_{CT}[\phi]$ is UV divergent.

The identification of the coefficients of the first few powers of ϕ on both sides gives

$$\begin{aligned} \partial_{\mu^2} \Gamma_0 &= 0, \\ \partial_{\mu^2} \Gamma_{2, x_1, x_2} &= \delta_{x_1, x_2}, \\ \partial_{\mu^2} \Gamma_{4, x_1, x_2, x_3, x_4} &= 0, \\ \partial_{\mu^2} \Gamma_{6, x_1, x_2, x_3, x_4, x_5, x_6} &= -\frac{\hbar}{2 \cdot 2^3} \sum_{\pi \in S_6} \text{Tr} [G_R \Gamma_{4 \ x_{\pi(1)}, x_{\pi(2)}} G_R \Gamma_{4 \ x_{\pi(3)}, x_{\pi(4)}} G_R \Gamma_{4 \ x_{\pi(5)}, x_{\pi(6)}} G_R], \end{aligned} \quad (18)$$

and

$$\begin{aligned} \partial_{\mu^2} \Gamma_{CT \ 0} &= \frac{\hbar}{2} \sum_{m=0}^{\infty} (-1)^m \text{Tr} [G_R (\Gamma_{CT \ 2} G_R)^m] = \frac{\hbar}{2} \text{Tr} (\Gamma_2 + \Gamma_{CT \ 2})^{-1}, \\ \partial_{\mu^2} \Gamma_{CT \ 2, x_1, x_2} &= \frac{\hbar}{4} \sum_{m=1}^{\infty} (-1)^m \sum_{\pi \in S_2} \sum_{j=1}^m \text{Tr} [G_R (\Gamma_{CT \ 2} G_R)^{j-1} G_R (\Gamma_{CT \ 2} G_R)^{m-j} (\Gamma_4 + \Gamma_{CT \ 4})_{x_{\pi(1)}, x_{\pi(2)}} G_R], \\ \partial_{\mu^2} \Gamma_{CT \ 4, x_1, x_2, x_3, x_4} &= \frac{\hbar}{8} \sum_{m=2}^{\infty} (-1)^m \sum_{\pi \in S_4} \sum_{j=1}^{m-2} \sum_{k=0}^{m-j-2} \text{Tr} [G_R (\Gamma_{CT \ 2} G_R)^{m-j-k-2} (\Gamma_4 + \Gamma_{CT \ 4})_{x_{\pi(1)}, x_{\pi(2)}} G_R \\ &\quad \times (\Gamma_{CT \ 2} G_R)^j (\Gamma_4 + \Gamma_{CT \ 4})_{x_{\pi(3)}, x_{\pi(4)}} G_R (\Gamma_{CT \ 2} G_R)^k] \\ &\quad + \frac{\hbar}{48} \sum_{m=1}^{\infty} (-1)^m \sum_{\pi \in S_4} \sum_{j=1}^m \\ &\quad \times \text{Tr} [G_R (\Gamma_{CT \ 2} G_R)^{j-1} G_R (\Gamma_{CT \ 2} G_R)^{m-j} (\Gamma_6 + \Gamma_{CT \ 6})_{x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}} G_R], \end{aligned} \quad (19)$$

for the first few coefficient functions.

C. Iterative solution

It is instructive to imagine the solution of the evolution equation as the fixed point of the iteration $\Gamma^{[N-1]}[\phi] \rightarrow \Gamma^{[N]}[\phi]$,

$$\Gamma_{\mu^2}^{[N+1]}[\phi] = \Gamma_{\mu_0^2}^{[N]}[\phi] + \frac{\hbar}{2} \int_{\mu_0^2}^{\mu^2} d\tilde{\mu}^2 \text{Tr} \left(\frac{\delta^2 \Gamma_{\tilde{\mu}^2}^{[N]}[\phi]}{\delta\phi\delta\phi} \right)^{-1} + \frac{\mu^2 - \mu_0^2}{2} \phi \cdot \phi. \quad (20)$$

The speed of convergence of the iterations depends on the choice of the starting point. Let us assume in the spirit of the perturbation expansion that the iteration reaches at least a fixed point when the tree-level starting point, given by Eq. (11), is used. The iteration in Eq. (20) becomes well-defined at each step only after imposing the initial condition (11) together with $\Gamma_{CT \mu_0^2}[\phi] = 0$ for the μ^2 -integration, i.e. the dressing is assumed to be completely suppressed at μ_0^2 .

For the sake of simplicity, we apply the local potential approximation, i.e. we restrict the field variable to be homogeneous, $\phi_x \rightarrow \Phi$, in the effective action which will be written as

$$\Gamma_{\mu^2}[\Phi] = V[U_{R \mu^2}(\Phi) + U_{CT \mu^2}(\Phi)]. \quad (21)$$

The starting point is a μ^2 -independent potential,

$$\begin{aligned} U_{R \mu^2}^{[0]}(\Phi) &= U_{\mu_0^2}(\Phi) = \frac{\mu_0^2}{2} \Phi^2 + \frac{g}{4!} \Phi^4, \\ U_{CT \mu^2}^{[0]}(\Phi) &= 0, \end{aligned} \quad (22)$$

and the iteration (20) becomes an integral equation,

$$\begin{aligned} U_{R \mu^2}^{[N+1]}(\Phi) &= \frac{\mu^2}{2} \Phi^2 + \frac{g}{4!} \Phi^4 + B_{R \mu_0^2 \rightarrow \mu^2}[U_{R \mu^2}^{(N)}], \\ U_{CT \mu^2}^{[N+1]}(\Phi) &= U_{CT \mu_0^2}^{[N]} + B_{CT \mu_0^2 \rightarrow \mu^2}[U_{R \mu^2}^{[N]}, U_{CT \mu^2}^{[N]}] \end{aligned} \quad (23)$$

for the local potentials which is written in $d = 4$ as

$$\begin{aligned} B_{R \mu_0^2 \rightarrow \mu^2}[U_{R \mu^2}] &= -\frac{\hbar}{2} \int_{\mu_0^2}^{\mu^2} d\tilde{\mu}^2 \int_p \left[\frac{1}{p^2 + U_{R \tilde{\mu}^2}''(\Phi)} - (1 - V_{R \tilde{\mu}^2}''(\Phi) G_{R \tilde{\mu}^2} + [V_{R \tilde{\mu}^2}''(\Phi) G_{R \tilde{\mu}^2}]^2) G_{R \tilde{\mu}^2} \right] \\ &= -\frac{\hbar}{2} \int_{\mu_0^2}^{\mu^2} d\tilde{\mu}^2 \sum_{m=3}^{\infty} \int_p [(V_{R \tilde{\mu}^2}''(\Phi) G_{R \tilde{\mu}^2})^m G_{R \tilde{\mu}^2}], \\ B_{CT \mu_0^2 \rightarrow \mu^2}[U_{R \mu^2}, U_{CT \mu^2}] &= -\frac{\hbar}{2} \int_{\mu_0^2}^{\mu^2} d\tilde{\mu}^2 \int_p \left[\frac{1}{p^2 + U_{R \tilde{\mu}^2}''(\Phi) + U_{CT \tilde{\mu}^2}''(\Phi)} - \frac{1}{p^2 + U_{R \tilde{\mu}^2}''(\Phi)} \right. \\ &\quad \left. + (1 - V_{R \tilde{\mu}^2}''(\Phi) G_{R \tilde{\mu}^2} + [V_{R \tilde{\mu}^2}''(\Phi) G_{R \tilde{\mu}^2}]^2) G_{R \tilde{\mu}^2} \right] \\ &= -\frac{\hbar}{2} \int_{\mu_0^2}^{\mu^2} d\tilde{\mu}^2 \int_p [1 - V_{R \tilde{\mu}^2}''(\Phi) G_{R \tilde{\mu}^2} + [V_{R \tilde{\mu}^2}''(\Phi) G_{R \tilde{\mu}^2}]^2] G_{R \tilde{\mu}^2} \\ &\quad - \frac{\hbar}{2} \int_{\mu_0^2}^{\mu^2} d\tilde{\mu}^2 \int_p \sum_{m=1}^{\infty} (-1)^m \frac{1}{p^2 + U_{R \tilde{\mu}^2}''(\Phi)} \left(\frac{U_{CT \tilde{\mu}^2}''(\Phi)}{p^2 + U_{R \tilde{\mu}^2}''(\Phi)} \right)^m, \end{aligned} \quad (24)$$

with the help of the renormalized propagator

$$G_{R \tilde{\mu}^2} = \frac{1}{p^2 + U_{R \tilde{\mu}^2}''(0)} \quad (25)$$

and the subtracted potential

$$V_{R \mu^2}(\Phi) = U_{R \mu^2}(\Phi) - \frac{1}{2} U_{R \mu^2}''(0) \Phi^2. \quad (26)$$

The multiplicative factors \hbar in front of the one-loop integral on the right hand sides indicate that each iteration resums a successive order of the loop-expansion within the local potential approximation. The result of the first iteration is called independent mode approximation because the quantum fluctuations are treated independently. The main simplification in computing in this order comes from the fact that the initial action is independent of μ^2 ,

$$\begin{aligned}
U_{R\mu^2}^{[1]}(\Phi) &= \frac{\mu^2}{2}\Phi^2 + \frac{g}{4!}\Phi^4 + U^{1l}(\Phi; \mu^2, g) - U^{1l}(\Phi; \mu_0^2, g) + \frac{\hbar}{2} \int_{\mu^2}^{\mu_0^2} d\tilde{\mu}^2 \sum_{n=0}^2 (-1)^n \int_p \left(\frac{\frac{g}{2}\Phi^2}{p^2 + \tilde{\mu}^2} \right)^n \frac{1}{p^2 + \tilde{\mu}^2} \\
&= \frac{\mu^2}{2}\Phi^2 + \frac{g}{4!}\Phi^4 + U_R^{1l}(\Phi; \mu^2, g) - U_R^{1l}(\Phi; \mu_0^2, g) \\
U_{CT\mu^2}^{[1]}(\Phi) &= -\frac{\hbar}{2} \int_{\mu^2}^{\mu_0^2} d\tilde{\mu}^2 \sum_{n=0}^2 (-1)^n \int_p \left(\frac{\frac{g}{2}\Phi^2}{p^2 + \tilde{\mu}^2} \right)^n \frac{1}{p^2 + \tilde{\mu}^2} \\
&= \frac{\hbar}{2} \int_p \left[\ln \frac{p^2 + \mu^2}{p^2 + \mu_0^2} + \frac{g}{2}\Phi^2 \left(\frac{1}{p^2 + \mu^2} - \frac{1}{p^2 + \mu_0^2} \right) - \frac{1}{2} \left(\frac{g}{2}\Phi^2 \right)^2 \left(\frac{1}{(p^2 + \mu^2)^2} - \frac{1}{(p^2 + \mu_0^2)^2} \right) \right], \quad (27)
\end{aligned}$$

where

$$U^{1l}(\Phi; \mu^2, g) = \frac{\hbar}{2} \int_p \ln \left(p^2 + \mu^2 + \frac{g}{2}\Phi^2 \right) \quad (28)$$

stands for the one-loop effective potential of the model of mass μ and coupling constant g and

$$U_R^{1l}(\Phi; \mu^2, g) = U^{1l}(\Phi; \mu^2, g) - \frac{\hbar}{2} \int_p \left[\ln(p^2 + \mu^2) + \frac{g}{2}\Phi^2 \frac{1}{p^2 + \mu^2} - \frac{1}{2} \left(\frac{\frac{g}{2}\Phi^2}{p^2 + \mu^2} \right)^2 \right] \quad (29)$$

represents its UV finite, renormalized part. When the limit $\mu_0^2 \rightarrow \infty$ is made then the $\mathcal{O}(\Phi^2/\mu_0^2)$ terms become negligible and we find the standard result,

$$\begin{aligned}
U_{R\mu^2}^{[1]}(\Phi) &= \frac{\mu^2}{2}\Phi^2 + \frac{g}{4!}\Phi^4 + U_R^{1l}(\Phi; \mu^2, g), \\
U_{CT\mu^2}^{[1]}(\Phi) &= \frac{\hbar}{2} \int_p \left[\ln(p^2 + \mu^2) + \frac{g}{2}\Phi^2 \frac{1}{p^2 + \mu^2} - \frac{1}{2} \left(\frac{g}{2}\Phi^2 \right)^2 \frac{1}{(p^2 + \mu^2)^2} \right], \quad (30)
\end{aligned}$$

up to a constant in the renormalized potential. It is easy to see that the subsequent iterations preserve the UV finiteness of $U_R(\Phi)$.

It is worthwhile mentioning that the split of the iteration of Eq. (20) into a renormalized and a counterterm part,

$$\partial_{\mu^2} \Gamma_R^{[N+1]}[\phi] = \frac{\hbar}{2} \sum_{m=[\frac{d}{2}]+1}^{\infty} (-1)^m \text{Tr} \left[G_R \cdot \left(\sum_{n=2}^{\infty} \frac{1}{(2n-2)!} \Gamma_{2n\ x_3, \dots, x_{2n}}^{[N]} \phi_{x_3} \cdots \phi_{x_{2n}} \cdot G_R \right)^m \right] + \frac{1}{2} \phi \cdot \phi, \quad (31)$$

and

$$\begin{aligned}
\partial_{\mu^2} \Gamma_{CT}^{[N+1]}[\phi] &= \frac{\hbar}{2} \sum_{m=0}^{\infty} (-1)^m \text{Tr} G_R \left[\left(\Gamma_{CT\ 2}^{[N]} + \sum_{n=2}^{\infty} \frac{1}{(2n-2)!} (\Gamma_{2n\ x_3, \dots, x_{2n}}^{[N]} + \Gamma_{CT\ 2n\ x_3, \dots, x_{2n}}^{[N]}) \phi_{x_3} \cdots \phi_{x_{2n}} \right) G_R \right]^m \\
&\quad - \frac{\hbar}{2} \sum_{m=[\frac{d}{2}]+1}^{\infty} (-1)^m \text{Tr} G_R \left[\sum_{n=2}^{\infty} \frac{1}{(2n-2)!} \Gamma_{2n\ x_3, \dots, x_{2n}}^{[N]} \phi_{x_3} \cdots \phi_{x_{2n}} G_R \right]^m, \quad (32)
\end{aligned}$$

realizes a particular BPHZ renormalization scheme. In fact, $\Gamma_R^{[N]}[\phi]$ is a finite functional for each N and the counterterms in $\Gamma_{CT}^{[N]}[\phi]$ correspond to the full subtraction of the overall divergences at each order N which are defined by the expansion of the integrand of the loop integral, the trace on the right hand side of Eq. (32).

III. 2PI EFFECTIVE ACTION

Our goal in this section is to derive the evolution equation for the 2PI effective action for the model defined by the action (4). This is achieved by generalizing the standard procedure, presented briefly in section II, for the bilocal field

$\varphi_x \varphi_y$. For this end we introduce the generating functional

$$e^{\frac{1}{\hbar} \tilde{W}[J]} = \int \mathcal{D}[\varphi] e^{-\frac{1}{\hbar} S[\varphi] + \frac{1}{2\hbar} \varphi \cdot J \cdot \varphi}, \quad (33)$$

together with its Legendre transform, the effective action for the two-point function,

$$\mathcal{G}_{x,y} = \langle T(\varphi_x \varphi_y) \rangle = \frac{\delta \tilde{W}[J]}{\delta \frac{1}{2} J_{x,y}}, \quad (34)$$

which is given by

$$\tilde{\Gamma}[\mathcal{G}] = -\tilde{W}[J] + \frac{1}{2} \text{Tr } \mathcal{G}^t \cdot J. \quad (35)$$

The inverse Legendre transformation is based on the relation

$$\frac{\delta \tilde{\Gamma}[\mathcal{G}]}{\delta \mathcal{G}_{x,y}} = \frac{1}{2} J_{x,y}. \quad (36)$$

Since the source $J_{x,y}$ is symmetrical, so is $\mathcal{G}_{x,y}$. Another important symmetry, translational invariance is supposed to be recovered in the limit $J \rightarrow 0$, therefore the propagator in the vacuum is taken to be translation invariant, $\langle T(\varphi_x \varphi_y) \rangle_{J=0} = \mathcal{G}_{x-y} = \mathcal{G}_{y-x}$. For the sake of simplicity we shall assume that the bare action is always chosen in such a manner that there is no condensation, $\langle \varphi_x \rangle_{J=0} = 0$, as in Section II.

The relation

$$I_{(x_1, x_2), (y_1, y_2)} = \frac{1}{2} (\delta_{x_1, y_1} \delta_{x_2, y_2} + \delta_{x_1, y_2} \delta_{x_2, y_1}) = \frac{\delta \mathcal{G}_{x_1, x_2}}{\delta \mathcal{G}_{y_1, y_2}}, \quad (37)$$

cf. Eq. (A5), can be used to establish the identity

$$\frac{\delta^2 \tilde{W}[J]}{\delta \frac{1}{2} J \delta \frac{1}{2} J} = \left[\frac{\delta^2 \tilde{\Gamma}[\mathcal{G}]}{\delta \mathcal{G} \delta \mathcal{G}} \right]^{-1} : I = \left[\frac{\delta^2 \tilde{\Gamma}[\mathcal{G}]}{\delta \mathcal{G} \delta \mathcal{G}} \right]^{-1I}, \quad (38)$$

where the inverse $A^{-1I} \equiv A^{-1} : I$ with respect to the operator I was introduced. The matrix I acts on the two-particle subspace by projecting into the symmetrical subspace and plays the role of the identity operator for indistinguishable particles.

A. Evolution equation

We shall use the coupling constant g to control the RG evolution in internal space and let it go from zero to its physical value. The corresponding evolution of $\tilde{\Gamma}[\mathcal{G}]$ is governed by the evolution equation

$$\partial_g \tilde{\Gamma}[\mathcal{G}] = -\partial_g \tilde{W}[J] = e^{-\frac{1}{\hbar} \tilde{W}[J]} \int \mathcal{D}[\varphi] \partial_g S[\varphi] e^{-\frac{1}{\hbar} S[\varphi] + \frac{1}{2\hbar} \varphi \cdot J \cdot \varphi} = \frac{1}{4!} \int_x \left[\hbar \left(\frac{\delta^2 \tilde{\Gamma}[\mathcal{G}]}{\delta \mathcal{G} \delta \mathcal{G}} \right)_{(x,x), (x,x)}^{-1I} + (\mathcal{G}_{x,x})^2 \right]. \quad (39)$$

It is worthwhile mentioning that the onset of the condensation is signaled by the vanishing of the restoring forces to the equilibrium position, acting on the quantum fluctuations, i.e. by $\mathcal{G}_p^{-1} = 0$. Therefore, the exclusion of the condensation induces the restriction $\mathcal{G}_p^{-1} \neq 0$.

It is easy to understand the evolution equation for the functional $\tilde{W}[J]$ graphically. In fact, let us assume the form

$$\tilde{W}[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{W}_{n \ X_1, \dots, X_n} J_{X_1} \cdots J_{X_n}, \quad (40)$$

where the repeated double-indices $X_j = (x_j, y_j)$ are integrated over and the evolution equation reads as

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} \partial_g \tilde{W}_{n \ X_1, \dots, X_n} J_{X_1} \cdots J_{X_n} &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\hbar}{3!} \int_x \tilde{W}_{n+2 \ (x,x), (x,x), X_1, \dots, X_n} J_{X_1} \cdots J_{X_n} \\ &+ \frac{1}{3!} \int_x \sum_{m,n=0}^{\infty} \frac{1}{m! n!} \tilde{W}_{m+1 \ (x,x), X_1, \dots, X_m} J_{X_1} \cdots J_{X_m} \tilde{W}_{n+1 \ (x,x), Y_1, \dots, Y_n} J_{Y_1} \cdots J_{Y_n}. \end{aligned} \quad (41)$$

Consider the contribution $\mathcal{O}(J^k)$ on both sides. On the left hand side we find ∂_g acting on the Green function with k pairs of field variables which is connected as far as the cutting of two particle lines are concerned. This Green function is the sum of Feynman graphs and the dependence on g appears through the factors g multiplying the vertices. Therefore, the right hand side is supposed to be the sum of these contributions. We turn now the four internal lines attached to the vertex on which ∂_g acts into external lines and consider the left-over graph. The first and the second terms on the right hand side stand for graphs whose left-over graph remains two-particle connected or falls into two disconnected components, respectively.

Motivated by the Hartree-type optimization of the truncated functional Callan-Symanzik equation presented in section II we split off a quadratic term from the effective action,

$$\tilde{\Gamma}[\mathcal{G}] = \Gamma[\mathcal{G}] + \frac{1}{2}\alpha_{X_1, X_2}\mathcal{G}_{X_1}\mathcal{G}_{X_2}, \quad (42)$$

where α denotes a g -dependent coefficient function which is vanishing for $g = 0$. The corresponding evolution equation is

$$\partial_g \Gamma[\mathcal{G}] = \frac{\hbar}{4!} \int_x \left[\frac{\delta^2 \Gamma[\mathcal{G}]}{\delta \mathcal{G} \delta \mathcal{G}} + \alpha \right]_{(x,x),(x,x)}^{-1I} + \mathcal{G} : \left(\frac{1}{4!} L - \frac{1}{2} \partial_g \alpha \right) : \mathcal{G}, \quad (43)$$

with $L_{(x,y),(u,v)} = \delta_{x,y} \delta_{u,v} \delta_{x,u}$. It is furthermore useful to separate the effects of interactions in the effective action by means of the parametrization

$$\Gamma[\mathcal{G}] = \Gamma^{free}[\mathcal{G}] + U[\mathcal{G}], \quad (44)$$

where the first term corresponds to the free theory, cf. Appendix B,

$$\Gamma^{free}[\mathcal{G}] = \frac{\hbar}{2} \text{Tr} \left(\mathcal{G} \cdot \frac{\delta^2 S[0]}{\delta \phi \delta \phi} \right) - \frac{\hbar}{2} \text{Tr} \ln \mathcal{G}, \quad (45)$$

and the ‘potential’ $U[\mathcal{G}]$ stands for the effects of the interactions. The initial condition, $U[\mathcal{G}] = 0$, for the evolution equation is imposed at $g = 0$.

B. Expansion around the vacuum

We return now to the evolution equation (43) which cannot be solved exactly. Its usefulness depends on the possibility of finding a good enough truncation, a projection into a restricted functional space. This latter should be, on the one hand, simple enough that the truncated equations can be solved either analytically or numerically and on the other hand, rich enough to incorporate the important effective vertices of the theory. The natural procedure is the functional Taylor expansion of the type Eq. (40) for $U[\mathcal{G}]$, truncated at order N , i.e. we retain the N -particle effective interactions. In order to render the Taylor series faster converging and to recover the 2PI vertex functions the Taylor expansion will be organized around the vacuum by splitting the propagator as $\mathcal{G} = \mathcal{G}_v + \mathcal{H}$ where \mathcal{G}_v is the propagator in the vacuum and writing the effective action as

$$\Gamma[\mathcal{H}] = \Gamma^{free}[\mathcal{G}_v + \mathcal{H}] + \gamma[\mathcal{H}], \quad (46)$$

with

$$\gamma(\mathcal{H}) = U[\mathcal{G}_v + \mathcal{H}] = \sum_{n=0}^N \frac{1}{n!} \gamma_{n \ X_1, \dots, X_n} \mathcal{H}_{X_1} \cdots \mathcal{H}_{X_n}. \quad (47)$$

The coefficient functions are symmetrical with respect to the exchange of any pairs of their variables, $\gamma_{n, \dots, X_j, \dots, X_k, \dots} = \gamma_{n, \dots, X_k, \dots, X_j, \dots}$ and are translation invariant because they correspond to the functional Taylor expansion around the translation invariant vacuum. This justifies the notation

$$\gamma_{n, (p_1, q_1), \dots, (p_n, q_n)} = \delta_{p_1+q_1+\dots+p_n+q_n, 0} \hat{\gamma}_{n, (p_1, q_1), \dots, (p_n, q_n)} \quad (48)$$

in momentum space. The functional Taylor expansion for the full effective action thus reads as

$$\Gamma[\mathcal{H}] = \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma_{n \ X_1, \dots, X_n} \mathcal{H}_{X_1} \cdots \mathcal{H}_{X_n}, \quad (49)$$

where the first few coefficients are

$$\begin{aligned}
\Gamma_0 &= \frac{\hbar}{2} \text{Tr} \left(\mathcal{G}_v \cdot \frac{\delta^2 S[0]}{\delta \phi \delta \phi} \right) - \frac{\hbar}{2} \text{Tr} \ln \mathcal{G}_v + \hat{\gamma}_0 \\
\Gamma_{1;(x,y)} &= \frac{\hbar}{2} \frac{\delta^2 S[0]}{\delta \phi_y \delta \phi_x} - \frac{\hbar}{2} \mathcal{G}_v^{-1}{}_{y,x} + \hat{\gamma}_{1;(x,y)}, \\
\Gamma_{2,X_1,X_2} &= \tilde{\mathcal{G}}_v^{I-1}{}_{X_1,X_2} + \hat{\gamma}_{2,X_1,X_2}, \\
\Gamma_{3;(x_1,y_1),(x_2,y_2),(x_3,y_3)} &= -\frac{\hbar}{2 \cdot 3!} \sum_{\pi \in S_3} \mathcal{G}_v^{-1}{}_{(y_{\pi(1)}, x_{\pi(2)})} \mathcal{G}_v^{-1}{}_{(y_{\pi(2)}, x_{\pi(3)})} \mathcal{G}_v^{-1}{}_{(y_{\pi(3)}, x_{\pi(1)})} + \hat{\gamma}_{3;(x_1,y_1),(x_2,y_2),(x_3,y_3)}, \\
\Gamma_{4;(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)} &= \frac{\hbar}{2 \cdot 4!} \sum_{\pi \in S_4} \mathcal{G}_v^{-1}{}_{(y_{\pi(1)}, x_{\pi(2)})} \mathcal{G}_v^{-1}{}_{(y_{\pi(2)}, x_{\pi(3)})} \mathcal{G}_v^{-1}{}_{(y_{\pi(3)}, x_{\pi(4)})} \mathcal{G}_v^{-1}{}_{(y_{\pi(4)}, x_{\pi(1)})} \\
&\quad + \hat{\gamma}_{4;(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)},
\end{aligned} \tag{50}$$

where

$$\tilde{\mathcal{G}}_{(x,y),(u,v)}^{I-1} = \tilde{\mathcal{G}}_{(x,y),(u,v)}^{-1} = \frac{\delta^2 \Gamma^{free}[\mathcal{G}]}{\delta \mathcal{G}_{x,y} \delta \mathcal{G}_{u,v}} = \frac{\hbar}{4} (\mathcal{G}_{v,x}^{-1} \mathcal{G}_{y,u}^{-1} + \mathcal{G}_{v,y}^{-1} \mathcal{G}_{x,u}^{-1}), \tag{51}$$

see Appendix C. The propagator in the vacuum satisfies the equation

$$\mathcal{G}_v^{-1} = \frac{\delta^2 S[0]}{\delta \phi \delta \phi} + \Sigma, \tag{52}$$

where the self energy is given by

$$\Sigma = 2\alpha : \mathcal{G}_v + 2\hat{\gamma}_1. \tag{53}$$

The evolution equation can now be written as

$$\begin{aligned}
\partial_g \Gamma &= \frac{\hbar}{4!} \sum_{m=0}^{\infty} (-1)^m \int_x \left\{ K : \left[\left(\sum_{n=3}^{\infty} \frac{1}{(n-2)!} \Gamma_{n;Z_3,\dots,Z_n} \mathcal{H}_{Z_3} \cdots \mathcal{H}_{Z_n} + \alpha \right) : K \right]^m \right\}_{(x,x),(x,x)} \\
&\quad + \mathcal{G} : \left(\frac{1}{4!} L - \partial_g \alpha \right) : \mathcal{G} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \Gamma_{n;Z_1,Z_2,\dots,Z_n} \mathcal{H}_{Z_2} \cdots \mathcal{H}_{Z_n} \partial_g \mathcal{G}_v{}_{Z_1} + \mathcal{G}_v : \alpha : \partial_g \mathcal{G}_v,
\end{aligned} \tag{54}$$

by using \mathcal{H} as an independent variable and suppressing the first two double-indices of the coefficient functions Γ_n . Here

$$K_{X,Y}^{-1} = \tilde{\mathcal{G}}_v^{I-1}{}_{X,Y} + \hat{\gamma}_{2,X,Y} \tag{55}$$

stands for the inverse of the two-particle propagator and the separated $\mathcal{O}(\mathcal{G}^2)$ term in the effective action is treated by the expansion. The identification of the coefficients of different powers of \mathcal{H} on the two sides up to $\mathcal{O}(\mathcal{H}^2)$ yields

$$\begin{aligned}
\partial_g \hat{\gamma}_0 &= \frac{\hbar}{4!} \int_x K_{(x,x),(x,x)} + \frac{\hbar}{4!} \int_x [K : (\alpha : K)^m]_{(x,x),(x,x)} + \mathcal{G}_v : \left(\frac{1}{4!} L - \frac{1}{2} \partial_g \alpha \right) : \mathcal{G}_v + (\Gamma_1 + \mathcal{G}_v : \alpha) : \partial_g \mathcal{G}_v, \\
\partial_g \gamma_{1,X} &= -\frac{\hbar}{4!} \hat{K}_{Z_1,Z_2} \Gamma_{3;Z_2,Z_1,X} + \frac{\hbar}{4!} \text{Tr} \hat{K} : (\alpha : K : \Gamma_{3;X} + \Gamma_{3;X} : K : \alpha) \\
&\quad + \frac{1}{3!} L_{X,Z} \mathcal{G}_v{}_Z - \partial_g \alpha_{X,Z} \mathcal{G}_v{}_Z + (\Gamma_{2;X,Z} + \alpha_{X,Z}) \partial_g \mathcal{G}_v{}_Z, \\
\partial_g \hat{\gamma}_{2,X_1,X_2} &= \frac{\hbar}{4!} \sum_{\pi \in S_2} \hat{K}_{Z_1,Z_2} (\Gamma_{3;Z_2,Z_3,X_{\pi(1)}} K_{Z_3,Z_4} \Gamma_{3;Z_4,Z_1,X_{\pi(2)}} - \Gamma_{4;Z_2,Z_1,X_{\pi(1)},X_{\pi(2)}}) \\
&\quad - \partial_g \alpha_{X_1,X_2} + \Gamma_{3,X_1,X_2,Z} \partial_g \mathcal{G}_v{}_Z,
\end{aligned} \tag{56}$$

with

$$\hat{K}_{X,Y} = \int_u K_{X,(u,u)} K_{(u,u),Y}. \tag{57}$$

C. Iterative solution

We follow the general strategy of Section II and rewrite the evolution equation (43) as integral equation, used for iteration,

$$\Gamma_g^{[N+1]}[\mathcal{G}] = \Gamma_{g=0}^{[N]}[\mathcal{G}] + \frac{\hbar}{4!} \int_0^g dg' \text{Tr } L : \left(\frac{\delta^2 \Gamma_{g'}^{[N]}[\mathcal{G}]}{\delta \mathcal{G} \delta \mathcal{G}} + \alpha_{g'} \right)^{-1I} + \mathcal{G} : \left(\frac{g}{4!} L - \frac{1}{2} \alpha_g \right) : \mathcal{G}. \quad (58)$$

We choose $\alpha = gaL/12$ with a g -independent constant a in the Hartree-term which guides the evolution and seek the effective action in the form

$$\Gamma_g[\mathcal{G}] = \Omega_g[\mathcal{G}] + \frac{g(1-a)}{4!} \mathcal{G} : L : \mathcal{G}. \quad (59)$$

The iteration of the functional $\Omega_g[\mathcal{G}]$,

$$\Omega_g^{[N+1]}[\mathcal{G}] = \Omega_{g=0}^{[N]}[\mathcal{G}] + B[\Omega_g^{[N]}], \quad (60)$$

is realized by

$$B[\Omega_g] = \frac{\hbar}{4!} \int_0^g dg' \text{Tr } L : \left[\frac{\delta^2 \Omega_{g'}[\mathcal{G}]}{\delta \mathcal{G} \delta \mathcal{G}} + \frac{g'a}{12} L \right]^{-1I}. \quad (61)$$

The starting point is the g -independent effective action $\Omega_g^{[0]} = \Gamma^{free}$ and its first iteration,

$$B[\Omega_g^{[0]}] = \frac{\hbar}{2} \text{Tr} \left\{ \left[\ln \left(\tilde{\mathcal{G}}^{I-1} : L^{-1} + \frac{ga}{12} \right) - \ln(\tilde{\mathcal{G}}^{I-1} : L^{-1}) \right] : I \right\}, \quad (62)$$

yields

$$\Omega_g^{[1]}[\mathcal{G}] = \Gamma^{free}[\mathcal{G}] + \frac{\hbar}{2} \text{Tr} \left[I : \ln \left(\mathbb{1} + \frac{ga}{12} L : \tilde{\mathcal{G}} \right) \right]. \quad (63)$$

The expansion of the logarithm on the right hand side gives

$$\Gamma_g^{[1]}[\mathcal{G}] = \Gamma^{free}[\mathcal{G}] - \frac{\hbar}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{-ga}{12} \right)^n \text{Tr} [(L : \tilde{\mathcal{G}})^n] + \frac{g(1-a)}{4!} \mathcal{G} : L : \mathcal{G}, \quad (64)$$

since $I : L = L$. This effective action corresponds to the sum of ring diagrams where the two-particle propagators are joined by the vertices g .

For the choice $a = 1$ the right hand side of Eq. (58) is $\mathcal{O}(\hbar)$ and each iteration resums a successive order of the loop expansion to the effective action, c.f. Eq. (53). This resummation explains the infinite series in $\Gamma_g^{[1]}[\mathcal{G}]$. When $a = 0$ is chosen then the right hand side of Eq. (58) is $\mathcal{O}(g)$ and each iteration resums an order of the perturbation expansion in g only. For any other values of a an effective interaction strength is used in the partial resummation of the loop expansion (second terms on the right hand sides of Eqs. (58) and (64)) which is corrected order-by-order in the perturbation expansion (third terms on the right hand sides of Eqs. (58) and (64)).

IV. RELATIONS BETWEEN FUNCTIONAL EQUATIONS

We comment in this Section first the relation of the evolution equations with well-known, non-perturbative iterative methods and next the similarity of the derivation leading to the evolution equations for the 1PI and 2PI Green functions, Eqs. (14) and (44), respectively.

The evolution equations, Eqs. (9) and (39) represent exact relations for the generating functionals which, assuming the proper initial conditions, lead to the determination of the 1PI and 2PI Green functions, respectively. It is natural to raise the question about the relation of these equations to the Schwinger-Dyson functional equation which is another exact equation satisfied by the Green functions. Both share a common mathematical origin, they express the way the path integral for the transition amplitudes transforms during the change of certain variables [22]. The Schwinger-Dyson equation, overviewed briefly in Appendix D, is a hierarchical set of equations as the evolution equations. The

Schwinger-Dyson equation for the two and four point irreducible functions, Eq. (D14) and (D21), could be converted into an evolution equation by acting on them by the operator ∂_λ . But the difference between the resulting equations and Eqs. (9) and (39) is that the formers contain the derivative of the Green functions with respect to the control parameter in an implicit manner. The evolution equation scheme is organized in such a manner that the derivatives are all explicitly expressed in terms of the Green functions themselves.

There are a few advantages when the evolution equation method is used. One problem with the Schwinger-Dyson equations is that it is not clear if its solution is unique. Their iteration may have different basin of attraction and the identification of the physical solution might be rather difficult. As long as the initial conditions are properly given in a nonphysical, but perturbative domain, the evolution equation method produces a unique solution.

Let us now compare the evolution equations for the 1PI and 2PI functions more carefully. A remarkable property of the evolution equation (9) is that it does not lead out from the space of quadratic functionals. Due to this reason it is essential to keep the interaction term in the initial condition. Another way to characterize this feature is to notice that the structure of the evolution equation reflects the choice of the μ^2 -dependent part of the bare action only. The information about the bare interaction is contained in the initial conditions to the evolution equation. The 2PI evolution equation, Eq. (39), on the contrary reflects the $\mathcal{O}(\varphi^4)$ contact interactions only and is independent of the $\mathcal{O}(\varphi^2)$ part of the bare action. In general, the evolution equation for the n -linear expression of the local field variable ϕ_x contains the $\mathcal{O}(\phi^{n+1})$ and higher order pieces of the bare action whose terms up to $\mathcal{O}(\phi^n)$ appear in the initial conditions only. An important consequence of this structure is the UV finiteness of Eq. (39). In fact, the integrand of the formal one-loop integral on the right hand side is given in terms of the independent variable \mathcal{G} and as it stands, Eq. (39) is a formal expression and recovers the well-known divergences for the choice $\mathcal{G}_p = \mathcal{O}(p^{-2})$ only. The counterterms to the 2PI effective action depend on the choice of the independent variable \mathcal{G} . It was due to such a complicated renormalization that we left the discussion of the 2PI evolution equation on the bare level in Section III and counterterms like in Section II were not introduced. As of the initial conditions are concerned, they must fix that part of the effective action which is not represented in the evolution equation, in particular the bare interactions or the free part in the 1PI and 2PI cases, respectively.

Finally, the similarity between the 1PI and 2PI evolution equations represents an opportunity to find an alternative to the matching strategy for the introduction of effective theories. This similarity suggests to replace the composite operator $\varphi_x \varphi_y$ by a bilocal effective field $\psi_{x,y}$ and inquire about the corresponding effective theory given by the action $S_{\text{eff}}[\psi]$. The construction of an effective theory for local composite operators is well established by means of the perturbation expansion [23], or of non-perturbative scheme [24]. Our starting point for the derivation of the effective theory for the bilocal field $\psi_{x,y}$ is the generating functional $\tilde{W}[J]$ of Eq. (33) for the connected Green functions of the composite operator $\varphi_x \varphi_y$. The free bare effective action, $S_{\text{eff}}^{(0)}[\psi]$, is chosen in such a manner that it reproduces the free propagation of two identical particles controlled by $\psi_{x,y}$,

$$\int \mathcal{D}[\varphi] e^{-\frac{1}{\hbar} \int_x [-\frac{1}{2} \varphi_x \square_x \varphi_x + \frac{1}{2} m^2 \varphi_x^2] + \frac{1}{2\hbar} \varphi \cdot J \cdot \varphi} = \int \mathcal{D}[\psi] e^{-\frac{1}{\hbar} S_{\text{eff}}^{(0)}[\psi] + \frac{1}{2\hbar} \psi \cdot J}. \quad (65)$$

The interactions can be incorporated by adding the piece $g \int_x \psi_{x,x}^2 / 4!$ to the effective action. The equivalence of the interactive theories, expressed by the relation

$$e^{\frac{1}{\hbar} \tilde{W}[J]} = \int \mathcal{D}[\psi] e^{-\frac{1}{\hbar} S_{\text{eff}}^{(0)}[\psi] - \frac{g}{4! \hbar} \int_x \psi_{x,x}^2 + \frac{1}{2\hbar} \psi \cdot J}, \quad (66)$$

can be verified by acting with the differential operator

$$e^{-\frac{\hbar g}{4!} \int_x \frac{\delta^2}{\delta \frac{1}{2} J_{x,x} \delta \frac{1}{2} J_{x,x}}} \quad (67)$$

on both sides of Eq. (65).

The first step of this argument, the construction of the free action $S_{\text{eff}}^{(0)}[\psi]$ is far more complicated than the second step, the inclusion of the interactions. This is not by accident, the non-interacting dynamics of equivalent particles has to include the symmetrization/antisymmetrization of the quantum states. This step introduces long range, topological correlations among the otherwise non-interacting particles. These correlations represent a far more difficult problem than the local interactions. This difficulty is particularly explicit in the construction of the density functional [16] for non-interacting particles when the effective action has strongly non-local vertices of infinitely high order without natural small parameter to justify any truncation. When the non-interacting particles are fermions this procedure amounts to bosonisation.

The difficulty of obtaining the free effective action is avoided when one uses the evolution equation method and pays the price of having an effective action for a bilocal variable $\mathcal{G}_{x,y}$ rather than the density functional which depends

on the local, diagonal part $\mathcal{G}_{x,x}$ only. The complications mentioned are then hidden in the logarithmic function of the initial condition Eq. (45).

The generalization of this construction for higher order composite operators is possible, as well. The only condition is that the interaction Lagrangian for the elementary fields must be expressible in terms of the composite fields. The difficult question, the issue of the choice of the interactions in the effective theory can be hidden in the initial conditions for the effective action for the composite fields. This latter can partially be taken over from the elementary theory without interactions (aspects of the statistics) and partially be generated by the evolution equation (aspects of the interactions) which provides an iterative improvement of the effective theory (cf. Section III C).

V. SUMMARY

The functional renormalization group strategy was studied in this paper in the internal space in the framework of the scalar ϕ^4 model where either the mass or the coupling strength can be evolved. The mass-controlled suppression of the amplitude of the fluctuations realizes the functional generalization of the Callan-Symanzik scheme for the bare Green functions. The appropriate split of the evolution equation into the sum of two pieces provides a non-perturbative BPHZ-like renormalization of the 1PI Green functions.

The control of the amplitude of the fluctuations by the interactions leads to the evolution equation generated by the change of the coupling strength. This scheme was exploited in order to obtain the evolution equation for the 2PI effective action.

Finally, the similarity of the evolution equations in these schemes was used to construct an effective bilocal theory for the connected propagator.

The advantage of the method, compared to the traditional external space renormalization group schemes, lies in its flexibility in choosing the parameter to evolve. This feature allows one to trace the dependence on certain scales in a model possessing several scales or on any of the coupling constants. This latter is important when symmetries prevent us to use momentum-space cutoff.

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APPENDIX A: NOTATION, CONVENTIONS

The space-time and Fourier-space integrals are defined as

$$\int_x = \int d^d x = a^d \sum_x, \quad \int_p = \int \frac{d^d p}{(2\pi)^d} = \sum_p, \quad f_x = \int_p f_p e^{ipx}, \quad (\text{A1})$$

and the Fourier transform of a translation invariant propagator $G_{x,y}$ is defined as $\delta_{p+q,0} G_p = G_{p,q}$. The composite index $X = (x, y)$, proves to be useful, e.g. $\int_{x,y} A_{x,y} B_{x,y} = \int_X A_X B_X$. The integration is frequently shown as scalar product, $f \cdot g = \int_x f_x g_x$, $A : B = \int_{x,y} F_{x,y} G_{x,y}$. The repeated indices are automatically summed/integrated over unless written explicitly otherwise.

The functional derivative is defined in d -dimensional lattice as

$$\frac{\delta}{\delta \phi_x} = \frac{1}{a^d} \frac{\partial}{\partial \phi_x^L} \quad (\text{A2})$$

where ϕ^L is the lattice field variable where the factor a^{-d} is needed in order to satisfy the relation

$$\frac{\delta}{\delta \phi_y} \int_x f(\phi_x) = f'(\phi_y). \quad (\text{A3})$$

In an analogous manner we have

$$\frac{\delta}{\delta \phi_p} = V \frac{\partial}{\partial \phi_p^L}. \quad (\text{A4})$$

Our convention for the derivative with respect to a symmetrical matrix \mathcal{G}^{sym} is

$$\frac{\delta}{\delta \mathcal{G}_{x,y}^{\text{sym}}} = \begin{cases} \frac{1}{2a^{2d}} \left(\frac{\partial}{\partial \mathcal{G}_{x,y}^{\text{non-sym}L}} + \frac{\partial}{\partial \mathcal{G}_{y,x}^{\text{non-sym}L}} \right) & x \neq y, \\ \frac{1}{a^{2d}} \frac{\partial}{\partial \mathcal{G}_{x,y}^{\text{sym}L}} & x = y, \end{cases} \quad (\text{A5})$$

where $\mathcal{G}^{\text{non-sym}}$ is a non-symmetrical matrix. It allows us the use of the chain rule

$$\frac{\delta}{\delta A_{x,y}} = \int_{u,v} \frac{\delta B_{u,v}}{\delta A_{x,y}} \frac{\delta}{\delta B_{u,v}}. \quad (\text{A6})$$

APPENDIX B: ONE-LOOP EFFECTIVE ACTION

The evolution equation for the 2PI effective action requires the knowledge of initial conditions imposed at vanishing or infinitesimal coupling strength. The loop-expansion for path integral Eq. (33) has two non-vanishing orders for $g = 0$, therefore these orders will be retained below.

1. $\mathcal{O}(\hbar^0)$

The generating functionals are given by

$$\begin{aligned} \tilde{W}^{(0)}[J] &= -S[\phi_0[J]] + \frac{1}{2} \phi_0[J] \cdot J \cdot \phi_0[J], \\ \tilde{\Gamma}^{(0)}[\mathcal{G}] &= S[\phi_0[J]] - \frac{1}{2} \phi_0[J] \cdot J \cdot \phi_0[J] + \frac{1}{2} \text{Tr } \mathcal{G}^t \cdot J, \end{aligned} \quad (\text{B1})$$

in the tree approximation where the stationary point $\phi_0[J]$ satisfies the equation

$$\frac{\delta S[\phi]}{\delta \phi} \Big|_{\phi=\phi_0[J]} = J \cdot \phi_0[J], \quad (\text{B2})$$

and the source J is related to the Green function \mathcal{G} via

$$\mathcal{G}_{x,y} = \phi_0 \, x[J] \phi_0 \, y[J], \quad (\text{B3})$$

i.e.

$$\phi_0 \, x[J] = \sqrt{\mathcal{G}_{x,x}}. \quad (\text{B4})$$

Notice that $\phi_0[J]$ is non-vanishing and is given by this equation in the phase with broken symmetry only. The elimination of the source J yields

$$\tilde{\Gamma}^{tree}[\mathcal{G}] = S[\phi_0[J]] = \int_x \left[-\frac{1}{8} (\ln \mathcal{G}_{x,x}) \square_x \mathcal{G}_{x,x} + \frac{\mu^2}{2} \mathcal{G}_{x,x} + \frac{g}{4!} (\mathcal{G}_{x,x})^2 \right]. \quad (\text{B5})$$

In the symmetrical phase without tree-level condensate $\tilde{\Gamma}^{tree}[\mathcal{G}] = 0$ holds.

2. $\mathcal{O}(\hbar)$

We write $\phi = \phi_0 + \eta$, expand the exponent of the integrand in powers of η keeping the terms up to $\mathcal{O}(\eta^2)$ and find

$$\tilde{W}^{(1)}[J] = -S[\phi_0[J]] + \frac{1}{2} \phi_0[J] \cdot J \cdot \phi_0[J] - \frac{1}{2} \hbar \text{Tr} \ln \left[\frac{\delta^2 S[\phi]}{\delta \phi \delta \phi} \Big|_{\phi_0[J]} - J \right]. \quad (\text{B6})$$

We define the connected two-point function G by

$$\mathcal{G}_{x,y} = \phi_0 \, x[J] \phi_0 \, y[J] + \hbar G_{x,y} = \frac{\delta W^{(1)}[J]}{\delta \frac{1}{2} J_{x,y}} = \phi_0 \, x[J] \phi_0 \, y[J] + \hbar \left[\frac{\delta^2 S[\phi]}{\delta \phi \delta \phi} \Big|_{\phi_0[J]} - J \right]_{y,x}^{-1} \quad (\text{B7})$$

which gives

$$G_{x,y} = \left[\frac{\delta^2 S[\phi]}{\delta \phi \delta \phi} \Big|_{\phi_0[J]} - J \right]_{y,x}^{-1}, \quad (\text{B8})$$

and

$$\tilde{\Gamma}^{(1)}[\mathcal{G}] = \Gamma^{tree}[\mathcal{G}] + \frac{\hbar}{2} \text{Tr} \ln G^{-1} + \frac{\hbar}{2} \text{Tr} \left[G^t \cdot \frac{\delta^2 S[\phi]}{\delta \phi \delta \phi} \right] \quad (\text{B9})$$

up to a \mathcal{G} -independent constant, c.f. Ref. [13]. In order to avoid the problem of non-convexity, we assume that the vacuum has no condensate, $\phi_0[J] = 0$, i.e. $\mathcal{G} = \hbar G$.

APPENDIX C: INVERSE OF SYMMETRIC 4-POINT FUNCTIONS

The quadratic part of the free effective action is

$$\tilde{\mathcal{G}}_{(x,y),(u,v)}^{I-1} = \frac{1}{4} \int_{p,q} \mathcal{G}_p^{-1} \mathcal{G}_q^{-1} \left(e^{ip(y-u)+iq(v-x)} + e^{ip(y-v)+iq(u-x)} \right), \quad (\text{C1})$$

for the translation invariant case. Therefore we seek the inversion of the operators of the form

$$A_{(x,y),(u,v)} = \frac{1}{2} \int_{p,q} A_{p,q} \left(e^{ip(x-u)+iq(y-v)} + e^{ip(x-v)+iq(y-u)} \right), \quad (\text{C2})$$

with symmetries $A_{(x,y),(u,v)} = A_{(u,v),(x,y)} = A_{(x,y),(v,u)}$ which require $A_{p,q} = A_{-q,-p} = A_{q,p}$. It is easy to see that A^{-1} cannot be written in the form (C2). But such operators can be inverted when they are acting on the space of symmetric matrices. In fact, the relation

$$\frac{1}{2} (\delta_{x,u} \delta_{y,v} + \delta_{x,v} \delta_{y,u}) = \frac{1}{2} \int_{p,q} A_{p,q} A_{p,q}^{-1} \left(e^{ip(y-v)+iq(u-x)} + e^{ip(y-u)+iq(v-x)} \right) \quad (\text{C3})$$

yields $I_{p,q} = 1$, $A_{p,q}^{-1I} = 1/A_{p,q}$ and

$$\tilde{\mathcal{G}}_{p,q}^{I-1} = \frac{1}{2} \mathcal{G}_p^{-1} \mathcal{G}_q^{-1}, \quad \tilde{\mathcal{G}}_{p,q} = 2\mathcal{G}_p \mathcal{G}_q \quad (\text{C4})$$

for the free two-particle propagator.

APPENDIX D: EQUATION OF MOTION

The Schwinger-Dyson equation is obtained in this Appendix for the two- and four-point 1PI functions. The absence of the condensate is assumed, in order to simplify the expressions.

1. Elementary field variable

The path integral

$$Z[j] \equiv e^{W[j]} = \int \mathcal{D}[\varphi] e^{-S[\varphi] + \varphi \cdot j} \quad (\text{D1})$$

yields the Schwinger-Dyson equation

$$\left[\frac{\delta S[\phi]}{\delta \phi} \Big|_{\phi = \frac{\delta}{\delta j}} - j \right] e^{W[j]} = 0, \quad (\text{D2})$$

which can be written as

$$\left[\frac{\delta S[\phi]}{\delta \phi} \Big|_{\phi = \frac{\delta}{\delta j} + \frac{\delta W[j]}{\delta j}} - j \right] \cdot \mathbf{1} = 0 \quad (\text{D3})$$

by means of the identity

$$\left[\frac{\delta}{\delta j}, e^{W[j]} \right] = \frac{\delta W[j]}{\delta j} e^{W[j]}. \quad (\text{D4})$$

For

$$S = \frac{1}{2} \varphi \cdot G_0^{-1} \cdot \varphi + \frac{g}{4!} \int_x \varphi^4 \quad (\text{D5})$$

Eq. (D3) yields the Schwinger-Dyson equation

$$0 = \phi_x + \frac{g}{6} \int_y G_0 \, x, y \left[\phi_y^3 + 3\phi_y \left(\frac{\delta^2 \Gamma[\phi]}{\delta \phi \delta \phi} \right)_{y,y}^{-1} + \frac{\delta^3 W[j]}{\delta j_y \delta j_y \delta j_y} \right] - (G_0 \cdot j)_x. \quad (\text{D6})$$

The first functional derivative of this equation is

$$\frac{\delta^2 W[0]}{\delta j_x \delta j_y} = G_0 \, x, y - \frac{g}{2} \int_z G_0 \, x, z \frac{\delta^2 W[0]}{\delta j_z \delta j_y} \frac{\delta^2 W[0]}{\delta j_z \delta j_z} - \frac{g}{6} \int_z G_0 \, x, z \frac{\delta^4 W[0]}{\delta j_z \delta j_z \delta j_z \delta j_y} \quad (\text{D7})$$

for $j = 0$. We shall express this relation by means of the 1PI vertex functions. By multiplying with G_0^{-1} and the inverse propagator we first arrive at

$$\frac{\delta^2 \Gamma[0]}{\delta \phi_x \delta \phi_y} = G_0^{-1} \, x, y + \frac{g}{2} \delta_{x,y} \frac{\delta^2 W[0]}{\delta j_x \delta j_x} + \frac{g}{6} \int_z \frac{\delta^4 W[0]}{\delta j_x \delta j_x \delta j_x \delta j_z} \frac{\delta^2 \Gamma[0]}{\delta \phi_z \delta \phi_y}. \quad (\text{D8})$$

In order to express the last term on the right hand side in terms of the effective action, one acts with the operator

$$\frac{\delta}{\delta j} = \frac{\delta \phi}{\delta j} \cdot \frac{\delta}{\delta \phi} = \frac{\delta^2 W[j]}{\delta j \delta j} \cdot \frac{\delta}{\delta \phi} = \left(\frac{\delta^2 \Gamma}{\delta \phi \delta \phi} \right)^{-1} \cdot \frac{\delta}{\delta \phi} \quad (\text{D9})$$

on the identity

$$\mathbb{1} = \frac{\delta^2 W[j]}{\delta j \delta j} \cdot \frac{\delta^2 \Gamma[\phi]}{\delta \phi \delta \phi} \quad (\text{D10})$$

and finds

$$0 = \int_z \frac{\delta^3 W[j]}{\delta j_{x_1} \delta j_{x_2} \delta j_z} \frac{\delta^2 \Gamma[\phi]}{\delta \phi_z \delta \phi_{x_3}} + \int_{\{z\}} \frac{\delta^2 W[j]}{\delta j_{x_1} \delta j_{z_1}} \frac{\delta^3 \Gamma[\phi]}{\delta \phi_{z_1} \delta \phi_{z_2} \delta \phi_{x_3}} \frac{\delta^2 W[j]}{\delta j_{z_2} \delta j_{x_2}}, \quad (\text{D11})$$

where the shorthand notation $\int_{\{z\}} = \int_{z_1, \dots, z_n}$ has been introduced. Multiplication by the second functional derivative of $W[j]$ in the index x_3 gives

$$\frac{\delta^3 W[j]}{\delta j_{x_1} \delta j_{x_2} \delta j_{x_3}} = - \int_{\{z\}} \frac{\delta^2 W[j]}{\delta j_{x_1} \delta j_{z_1}} \frac{\delta^2 W[j]}{\delta j_{x_2} \delta j_{z_2}} \frac{\delta^2 W[j]}{\delta j_{x_3} \delta j_{z_3}} \frac{\delta^3 \Gamma[\phi]}{\delta \phi_{z_1} \delta \phi_{z_2} \delta \phi_{z_3}}. \quad (\text{D12})$$

A further derivation leads to

$$\begin{aligned} \frac{\delta^4 W[j]}{\delta j_{x_1} \delta j_{x_2} \delta j_{x_3} \delta j_{x_4}} &= \int_{\{z\}} \left[- \frac{\delta^2 W[j]}{\delta j_{x_1} \delta j_{z_1}} \frac{\delta^2 W[j]}{\delta j_{x_2} \delta j_{z_2}} \frac{\delta^2 W[j]}{\delta j_{x_3} \delta j_{z_3}} \frac{\delta^2 W[j]}{\delta j_{x_4} \delta j_{z_4}} \frac{\delta^4 \Gamma[\phi]}{\delta \phi_{z_1} \delta \phi_{z_2} \delta \phi_{z_3} \delta \phi_{z_4}} \right. \\ &\quad + \frac{\delta^2 W[j]}{\delta j_{x_1} \delta j_{z_4}} \frac{\delta^2 W[j]}{\delta j_{x_2} \delta j_{z_5}} \frac{\delta^2 W[j]}{\delta j_{x_3} \delta j_{z_6}} \frac{\delta^3 \Gamma[\phi]}{\delta \phi_{z_4} \delta \phi_{z_5} \delta \phi_{z_6}} \frac{\delta^2 W[j]}{\delta j_{x_2} \delta j_{z_2}} \frac{\delta^2 W[j]}{\delta j_{x_3} \delta j_{z_3}} \frac{\delta^3 \Gamma[\phi]}{\delta \phi_{z_1} \delta \phi_{z_2} \delta \phi_{z_3}} \\ &\quad \left. + (x_1 \longleftrightarrow x_3) + (x_1 \longleftrightarrow x_4) \right] \end{aligned} \quad (\text{D13})$$

which finally allows us to write Eq. (D8) as

$$\begin{aligned} \frac{\delta^2 \Gamma[0]}{\delta \phi_x \delta \phi_y} &= G_{0 \ x, y}^{-1} + \frac{g}{2} \delta_{x, y} \left(\frac{\delta^2 \Gamma[0]}{\delta \phi \delta \phi} \right)_{x, x}^{-1} \\ &\quad - \frac{g}{6} \int_{\{z\}} \left(\frac{\delta^2 \Gamma[0]}{\delta \phi \delta \phi} \right)_{x, z_1}^{-1} \left(\frac{\delta^2 \Gamma[0]}{\delta \phi \delta \phi} \right)_{x, z_2}^{-1} \left(\frac{\delta^2 \Gamma[0]}{\delta \phi \delta \phi} \right)_{x, z_3}^{-1} \frac{\delta^4 \Gamma[0]}{\delta \phi_{z_1} \delta \phi_{z_2} \delta \phi_{z_3} \delta \phi_y} \\ &\quad + \frac{g}{2} \int_{\{z\}} \left(\frac{\delta^2 \Gamma[0]}{\delta \phi \delta \phi} \right)_{x, z_4}^{-1} \left(\frac{\delta^2 \Gamma[0]}{\delta \phi \delta \phi} \right)_{z_1, z_6}^{-1} \frac{\delta^3 \Gamma[0]}{\delta \phi_{z_4} \delta \phi_y \delta \phi_{z_6}} \left(\frac{\delta^2 \Gamma[0]}{\delta \phi \delta \phi} \right)_{x, z_2}^{-1} \left(\frac{\delta^2 \Gamma[0]}{\delta \phi \delta \phi} \right)_{x, z_3}^{-1} \frac{\delta^3 \Gamma[0]}{\delta \phi_{z_1} \delta \phi_{z_2} \delta \phi_{z_3}}. \end{aligned} \quad (\text{D14})$$

2. Product of two field variables

The path integral

$$\tilde{Z}[J] \equiv e^{\tilde{W}[J]} = \int \mathcal{D}[\varphi] e^{-S[\varphi] + \frac{1}{2} \varphi \cdot J \cdot \varphi} \quad (\text{D15})$$

leads to the relations

$$\begin{aligned} \frac{\delta \tilde{W}[0]}{\delta \frac{1}{2} J_{1,2}} &= \frac{\delta^2 W[0]}{\delta j_1 \delta j_2} = G_{1,2}^c = \mathcal{G}_{\mathbf{v} \ 1,2}, \\ \frac{\delta^2 \tilde{W}[0]}{\delta \frac{1}{2} J_{1,2} \delta \frac{1}{2} J_{3,4}} &= K_{(1,2),(3,4)}, \\ \frac{\delta^3 \tilde{W}[0]}{\delta \frac{1}{2} J_{1,2} \delta \frac{1}{2} J_{3,4} \delta \frac{1}{2} J_{5,6}} &= \frac{\delta^3 \tilde{Z}[0]}{\delta \frac{1}{2} J_{1,2} \delta \frac{1}{2} J_{3,4} \delta \frac{1}{2} J_{5,6}} - K_{(1,2),(3,4)} \mathcal{G}_{\mathbf{v} \ 5,6} - K_{(1,2),(5,6)} \mathcal{G}_{\mathbf{v} \ 3,4} - K_{(3,4),(5,6)} \mathcal{G}_{\mathbf{v} \ 1,2} \\ &\quad - \mathcal{G}_{\mathbf{v} \ 1,2} \mathcal{G}_{\mathbf{v} \ 3,4} \mathcal{G}_{\mathbf{v} \ 5,6}, \end{aligned} \quad (\text{D16})$$

where the two-particle propagator

$$K = \left(\frac{\delta^2 \tilde{\Gamma}[\mathcal{G}_{\mathbf{v}}]}{\delta \mathcal{G} \delta \mathcal{G}} \right)^{-1} \quad (\text{D17})$$

has been introduced. (For the sake of simplicity the space-time coordinates x_n are replaced by the corresponding indices n .) The analogue of Eq. (D12) is

$$\frac{\delta^3 \tilde{W}[J]}{\delta \frac{1}{2} J_{X_1} \delta \frac{1}{2} J_{X_2} \delta \frac{1}{2} J_{X_3}} = - \left(\frac{\delta^2 \tilde{\Gamma}[\mathcal{G}]}{\delta \mathcal{G} \delta \mathcal{G}} \right)_{X_1, Z_1}^{-1} \left(\frac{\delta^2 \tilde{\Gamma}[\mathcal{G}]}{\delta \mathcal{G} \delta \mathcal{G}} \right)_{X_2, Z_2}^{-1} \left(\frac{\delta^2 \tilde{\Gamma}[\mathcal{G}]}{\delta \mathcal{G} \delta \mathcal{G}} \right)_{X_3, Z_3}^{-1} \frac{\delta^3 \tilde{\Gamma}[\mathcal{G}]}{\delta \mathcal{G}_{Z_1} \delta \mathcal{G}_{Z_2} \delta \mathcal{G}_{Z_3}}. \quad (\text{D18})$$

Therefore, we have

$$\begin{aligned} G_{1,2}^c &= \mathcal{G}_{v \ 1,2}, \\ G_{1,2,3,4}^c &= K_{(1,2),(3,4)} - \tilde{\mathcal{G}}_{v \ (1,2),(3,4)} \\ G_{1,2,3,4,5,6}^c &= -K_{(1,2),Z_1} K_{(3,4),Z_2} K_{(5,6),Z_3} \frac{\delta^3 \Gamma[\mathcal{G}_v]}{\delta \mathcal{G}_{Z_1} \delta \mathcal{G}_{Z_2} \delta \mathcal{G}_{Z_3}} \\ &\quad - [K_{(1,2),(3,5)} G_{4,6}^c + K_{(1,2),(4,5)} G_{3,6}^c + K_{(1,2),(3,6)} G_{4,5}^c + K_{(1,2),(4,6)} G_{3,5}^c + G_{1,3,4,5}^c G_{2,6}^c + G_{2,3,4,5}^c G_{1,6}^c \\ &\quad + G_{1,3,4,6}^c G_{2,5}^c + G_{2,3,4,6}^c G_{1,5}^c + G_{1,3,5,6}^c G_{2,4}^c + G_{2,3,5,6}^c G_{1,4}^c + G_{1,4,5,6}^c G_{2,3}^c + G_{2,4,5,6}^c G_{1,3}^c] \end{aligned} \quad (\text{D19})$$

with $\tilde{\mathcal{G}}_{v \ X,Y}$ the free two-particle propagator. The first functional derivative of the Schwinger-Dyson equation can be written as

$$\Sigma_{x,y} = \frac{g}{2} \delta_{x,y} \mathcal{G}_{v \ x,x} + \frac{g}{6} \int_z (K_{(x,x),(x,z)} - \tilde{\mathcal{G}}_{v \ (x,x),(x,z)}) \mathcal{G}_{v \ z,y}^{-1}. \quad (\text{D20})$$

Notice that the leading order perturbative result is recovered for $K = \tilde{\mathcal{G}}_v$ only. When the two-particle sector is non-trivial by the choice of the independent variable \mathcal{G} then the one-particle sector of the theory is naturally modified.

In a similar manner the third functional derivative of the Schwinger-Dyson equation reads as

$$\begin{aligned} K_{(x,y),(u,v)} &= \tilde{\mathcal{G}}_{v \ (x,y),(u,v)} + \frac{g}{6} \int_z G_{0 \ x,z} \left(K_{(z,z),Z_1} K_{(z,y),Z_2} K_{(u,v),Z_3} \frac{\delta^3 \Gamma[\mathcal{G}_v]}{\delta \mathcal{G}_{Z_1} \delta \mathcal{G}_{Z_2} \delta \mathcal{G}_{Z_3}} \right. \\ &\quad \left. + K_{(z,z),(z,u)} \mathcal{G}_{v \ y,v} + K_{(z,z),(z,v)} \mathcal{G}_{v \ y,u} - K_{(z,z),(u,v)} \mathcal{G}_{v \ z,y} - K_{(z,y),(u,v)} \mathcal{G}_{v \ z,z} + \mathcal{G}_{v \ z,z} \tilde{\mathcal{G}}_{v \ (z,y),(u,v)} \right). \end{aligned} \quad (\text{D21})$$